Exercise 1 (10 + 15 points)

a) What is the DFT of \((1, 0, 0, 0) \in \mathbb{C}^4\)? And of which vector is \((1, 0, 0, 0)\) the DFT?

b) Apply RECURSIVE-FFT (version of the lecture or CLRS Chapter 30.2) to \((1, 0, 1, -1) \in \mathbb{C}^4\).

Provide the final output as well as all intermediate steps in the execution.

Exercise 2 (10 points)

Prove that \(n\) point-value pairs (with distinct points) are necessary to uniquely specify a polynomial of degree smaller than \(n\), that is show that for any distinct \(x_0, \ldots, x_{n-2} \in \mathbb{C}\) and any \(y_0, \ldots, y_{n-2} \in \mathbb{C}\) there exist at least two different polynomials \(A(x)\) and \(B(x)\) of degree smaller than \(n\) with

\[ A(x_j) = B(x_j) = y_j, \quad j = 0, \ldots, n-2. \]

Exercise 3 (30 points)

Let \(A(x) = \sum_{j=0}^{n-1} a_j x^j\) and \(x_0 \in \mathbb{C}\). Show that there exists a unique polynomial \(Q(x)\) of degree smaller than \(n-1\) and a unique \(r \in \mathbb{C}\) such that

\[ A(x) = Q(x) \cdot (x - x_0) + r. \]

Provide the pseudocode of an algorithm that computes the coefficients of \(Q(x)\) and \(r\) with running time \(\mathcal{O}(n)\).

Exercise 4 (5 + 30 points)

a) Given distinct \(x_0, \ldots, x_{n-1} \in \mathbb{C}\) and arbitrary \(y_0, \ldots, y_{n-1} \in \mathbb{C}\), the interpolation problem requires to find a polynomial \(P(x)\) of degree smaller than \(n\) such that \(P(x_j) = y_j, \ j = 0, \ldots, n-1\). Show that the solution to this problem is given by

\[ P(x) = \sum_{k=0}^{n-1} y_k \prod_{j \neq k} \left( x - x_j \right). \]

b) Provide a strategy that allows us to find the coefficients of \(P(x)\) as given in (1) with running time \(\mathcal{O}(n^2)\).

*Hint: Find the coefficients of \(\prod_{j \in \{0, \ldots, n-1\}} (x - x_j)\) and make use of Exercise 3. You may make use of Exercise 3 even if you have not solved it.*