Exercise 1 (10 points)
Demonstrate what happens when we insert the keys 5, 28, 19, 15, 20, 33, 12, 17, 10 (in this order) into a hash table with collisions resolved by chaining. Let the hash table have 9 slots and let the hash function be \( h(k) = k \mod 9 \).

Exercise 2 (15 points)
Assume we are storing a set of \( n \) keys into a hash table of size \( m \). Show that if the keys come from a universe \( U \) with \( |U| > nm \), then \( U \) has a subset of size \( n \) consisting of keys that all hash to the same slot, so that the worst-case searching time for hashing with chaining is \( \Theta(n) \).

Exercise 3 (15 + 10 points)
\[ a) \text{ Let } p, m \in \mathbb{N} \text{ with } p \text{ prime and } p > m > 1. \text{ Let } r \in \{0, 1, \ldots, p - 1\} \text{ and} \]
\[ S = \{ s \in \{0, 1, \ldots, p - 1\} : s \neq r \land s \equiv r \pmod{m} \}. \]
Show that \( |S| \leq \left\lfloor \frac{p}{m} \right\rfloor - 1 \).
\[ b) \text{ Let } a, b \in \mathbb{N}. \text{ Show that} \]
\[ \left\lceil \frac{a}{b} \right\rceil \leq \frac{a}{b} + \frac{b - 1}{b}. \]

Exercise 4 (50 points)
Let \( \mathcal{H} \) be a finite collection of hash functions from a given universe \( U \) of keys to \( \{0, 1, \ldots, m - 1\} \). We call \( \mathcal{H} \) universal if for all \( k_1, k_2 \in U \) with \( k_1 \neq k_2 \)
\[ \Pr_h [h(k_1) = h(k_2)] \leq \frac{1}{m}, \]
where the probability is over \( h \) chosen uniformly at random from \( \mathcal{H} \).
Consider \( U = \{0, 1\}^u \) for some \( u \in \mathbb{N} \) and let \( m = 2^b \) for some \( b \in \mathbb{N} \). For \( A \in \{0, 1\}^{b \times u} \) let \( h_A \) be
\[ h_A : U \to \{0, 1, \ldots, m - 1\}, \quad h_A(k) = \sum_{j=1}^{u} 2^{i-1} \left( \sum_{j=1}^{b} A_{ij} \cdot k_j \right) \mod 2, \]
where \( k = (k_1, k_2, \ldots, k_u) \). Show that \( h_A \neq h_B \) for \( A \neq B \) and that \( \mathcal{H} = \{h_A : A \in \{0, 1\}^{b \times u}\} \) is universal.